Modern Czech Logic: Vopěnka and Hájek, History and Background

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Abstract

We review some aspects of the history of modern Czech logic, with an emphasis on the role of Petr Vopěnka and Petr Hájek in the earlier stage of their careers.

Keywords: Petr Vopěnka, Petr Hájek, Set Theory, Nonstandard Numbers, Semiset.

1 Introduction

This paper is devoted to two distinguished Czech logicians, *Petr Hájek* and *Petr Vopěnka*. While Petr Hájek is well-known everywhere and has thousands of citations to his works in the WoS database, Petr Vopěnka is much less known. He did have some contacts in Poland and in Russia, but he did not travel much, and almost never left the Russia-dominated part of the world, and thus (as I was told by Petr Hájek) some of the people who knew his name even suspected that he might be a virtual person, much like Nicholas Bourbaki.

Both Hájek and Vopěnka had a major influence on modern Czech (and world) logic. In this paper we will review some of their results and achievements. We will also mention their motivations and some social aspects of their work, and try to say something about their philosophical background. Vopěnka, though he had fewer Ph.D. students and fewer citations than Hájek, was nonetheless a key figure in the history of logic. We think that his early work can serve as a subject matter of an interesting historical research project. This paper could be viewed as an early step in such research. However, what is written here is based more on personal memories than on serious study of sources.

*I would like to thank Bohuslav Balcar, Tomáš Jech, Peter Vojtáš, Derek von Barandy and Eva Ullrichová for useful comments, and for remarks that improve readability.

2 Some genealogy

Petr Hájek (1940–2016) was officially a student of *Ladislav Svante Rieger*, an algebraic logician who collaborated for example with Helena Rasiowa and Roman Sikorski. Among other things, Rieger invented the Rieger-Nishimura lattice (or Rieger-Nishimura ladder), a beautiful structure of infinitely many non-equivalent intuitionistic propositional formulas built up from a single atom: if the atom is p, then the bottom formulas in the ladder are $\bot$, p, $\neg p$, $p \vee \neg p$, $\neg\neg p$, $\neg\neg p \to p$, $\neg p \vee \neg\neg p$, ... The same structure was later independently invented by Nishimura. Rieger was also aware of the existence of nonstandard models of Peano arithmetic, which was by no means commonplace in the 1950s. Vopěnka was not sure about the details and origin of the proof known to Rieger, but thought that the construction was in fact Skolem's. A visitor of the Czech Republic may notice Rieger streets and Rieger park in Prague and in other cities; these are named after František Ladislav Rieger, who was active in the Czech national movement in the 19th century and was the great-grandfather of Rieger the mathematician.

Rieger (1916–1963) worked with both Vopěnka and Hájek, but he died soon after Hájek became his student. Hájek then collaborated with Vopěnka and never failed to mention that it was in fact Vopěnka who was his teacher.

Petr Vopěnka (1936–2015) would not call himself a logician: his original research interests, lasting in a sense through his entire life, were geometry and topology. He was a student (the last student) of Czech topologist *Eduard Čech*, who strongly influenced him and "showed him how to do mathematics". Čech is known by Čech cohomology, and some Czech mathematicians believe that it was he who invented Stone (or Čech-Stone) compactification. Vopěnka, when asked about it, once explained that as to Čech and Stone one cannot really tell which results should be attributed to whom: they worked together and were in friendly and frequent contact.

3 Vopěnka's set theory seminar

During the 1960s Vopěnka ran a seminar devoted to set theory. It was known as *Vopěnka's seminar* because Vopěnka was much more than an organizer. The seminar took place at the School of Mathematics and Physics of Charles University every Monday afternoon. Its participants usually waited to hear what Vopěnka was able to invent during the weekend and (I am told) during the late Sunday night, and he managed to stimulate research and suggest problems that they found meaningful.

The participants of the seminar were people who later became known as experts in set theory and in other fields, for example Bohuslav Balcar, Petr Hájek, Karel

Hrbáček, Tomáš Jech, Karel Příkrý, Antonín Sochor and Petr Štepánek. Jiří Polívka and Oswald Demuth were not really set theorists, but it is important to mention them, too. Demuth studied in Russia and worked (very successfully) in constructivist mathematics done in the Russian style. He probably did not attend Vopěnka's seminar (indeed, set theory was not consistent with his philosophical views) but was in contact with Vopěnka and can be counted among his group. Polívka was a philosopher. In those times Vopěnka was not only a respected leader of a research group—at the School of Mathematics he was also one of the decision makers. He established a new department and became its head, founded a field of study in which the students of mathematics could specialize (its official title was "theoretical cybernetics", but to large extent it was logic), and he had a say in hiring. He wanted to have a philosopher (i.e., Polívka) in his group, and had the power to make it happen. Two members of the group, Hájek and Sochor, were officially affiliated with the Czechoslovak Academy of Sciences, which in those times was a research institution without study programs, but with the right to award Ph.D. degrees. The communist regime became more and more liberal in the 1960s. Nevertheless, Hájek and Sochor could not work at the university because they were known to be active Christians (an Evangelical and a Catholic respectively). Hájek even served as an organist in his church. The different official employers had no impact on the research interests of Hájek and Sochor, who normally participated in the work of the seminar.

Hájek was familiar with the logical literature and probably played the role of a set theorist who knew a lot about logic. In general, he put emphasis on contacts with logicians from Europe and the U.S., could fluently communicate in several languages and was sure that results should be presented at conferences or sent to journals published in English. All this was less true about Vopěnka, who in general preferred thinking to reading or to attending conferences. This is not meant to say that Vopěnka was worse in communication: for me personally, he was the best speaker among the professors I met at the Mathematical School in the 1970s. However, he was an excellent speaker and teacher only when speaking Czech.

A big stimulus for Czech set theorists was the continuum problem, which was unsolved in the early sixties. Gödel's proof of the consistency of the continuum hypothesis CH, employing the constructible universe, was known to them, but they were also aware that quite different model constructions would be needed to show the *unprovability* of CH. Some other tools were at hand: Vopěnka for example published an ultraproduct construction of a model of the Gödel-Bernays set theory GB. Thus when Paul Cohen solved the continuum problem in 1963, Prague set theorists were not unprepared. Vopěnka seldom attended conferences, but in 1963 he accidentally was present (in Nice, France, or maybe in the U.S., but some of his colleagues think that it was in Vienna) at a meeting where Cohen presented his result. While Cohen probably considered his discovery a single-purpose proof and did not continue working on consistency results, people around Vopěnka started to think about turning Cohen's proof into a *method* that would yield further results. Such a method was established as soon as the following year, i.e. in 1964. It was called the ∇-*model* and it was a variant of the method of *Boolean valued models*. The same method was independently but somewhat later invented by Scott and Solovay.

Out of the axiomatic set theories, the Zermelo-Fraenkel set theory ZF is now more popular. However, Prague set theorists preferred to work with the Gödel-Bernays set theory GB, which is nowadays sometimes denoted NBG, where the letter 'N' refers to von Neumann. As the reader probably knows, the primary notion of ZF is *set*, and all other notions (number, function, ...) are reduced to it. Both theories are formulated in the same language $\{\in\}$ containing one unary predicate symbol for membership. Besides $\in$, the equality symbol $=$ can also appear in formulas (both theories are theories in predicate logic with equality). In GB, the primary notion is *class*, and sets are defined as those classes that are elements of other classes. It is common to use the uppercase letters X, X_1, Y, ... to denote classes, and the lowercase letters x, y, etc. to denote sets. The axiom system of GB is usually presented as several axioms (e.g. the axiom of extensionality) and one schema, the *comprehension schema*. A *formula* of GB is *normal* if the class variables are not quantified in it. Here are examples of normal formulas: $x \neq x$, $\neg\exists v(v \in x)$, $x \in Y_1 \;\&\; x \in Y_2$, $x \notin x$, and $x = x$. The comprehension schema stipulates that every normal formula $\varphi(x, z_1, .., z_k, Y_1, .., Y_r)$, with a dedicated variable x and any number of set parameters $z_1, .., z_k$ and class parameters $Y_1, .., Y_r$, determines a class (of all sets x such that $\varphi(x, \underline{z}, \underline{Y})$); this class is unique by the axiom of extensionality. The classes determined by the five example formulas above are the empty class $\emptyset$, the class of all empty sets (which is single-element or empty according to whether $\emptyset$ is a set), the intersection $Y_1 \cap Y_2$ of the classes Y_1 and Y_2, the class D of all sets that are not elements of themselves, and the universal class (of *all* sets, traditionally denoted V).

There is nothing paradoxical about the class D of all sets that are not elements of themselves; in GB, Russell's argument becomes a proof that D is not a set, i.e. that it is a *proper class*. The fact that a subclass of a set is again a set is provable in GB. Therefore, the universal class V is a proper class as well. Sets exist, and thus the empty class $\emptyset$, being a subclass of every class, is a set.

It is known that GB is conservative over ZF w.r.t. set sentences: any sentence not containing class variables is provable in GB if and only if it is provable in ZF. Therefore, ZF and GB can be seen as variants of the same theory that differ only inessentially. One can think that proper classes just simplify language in some cases. However, the relation between ZF and GB is quite involved, and actually the Prague group contributed to its clarification. One of their results will be outlined here.

A *truth relation* on a number n is a relation R between *set formulas* smaller than n and valuations of variables such that R satisfies Tarski's conditions (whenever applicable) with respect to the entire universe of sets. In more details, we can assume that variables that can appear in formulas are taken from a countably infinite set $\mathrm{Var} = \{v_0, v_1, v_2, \dots\}$ and that we have coding of formulas using natural numbers. That is, formulas can be identified with their numerical codes. Among numbers smaller than n some are (codes of) formulas, i.e. some are Gödel numbers (of formulas). A valuation of variables is any function defined on the set Var. If R is a truth relation on n, then a pair $[v_i \in v_j, e]$, where $v_i \in v_j$ is a formula smaller than n and e is a valuation, is in R if and only if $e(v_i)$ is an element of $e(v_j)$. A pair $[\varphi \,\&\, \psi, e]$, where again $\varphi \,\&\, \psi < n$ and e is a valuation, is in R if and only if both pairs $[\varphi, e]$ and $[\psi, e]$ are in R, and similarly for the atomic formula $v_i = v_j$ and for other logical connectives and quantifiers. One can think of a truth relation on n as a table with finitely many lines (in the sense of GB, i.e. according to the definition of finite set formulated in GB) corresponding to those numbers smaller than n that are formulas, and with class-many columns corresponding to all valuations. The following facts can be proved in GB about truth relations: if a truth relation on n exists, then it is unique, a truth relation on 0 does exist, and if there exists a truth relation on n, then there also exists a truth relation on $n+1$. A number is *occupable*, denoted $\mathrm{Ocp}(n)$, if there exists a truth relation on n. Occupable number is a well defined notion in GB. However, the formula $\mathrm{Ocp}(n)$ is not normal (because it starts with the class quantifier $\exists R$), and thus in GB it is not guaranteed that it determines a class. It is however clear that if it determined a class, then the class would in fact be a set because it would be a subclass of the set of all natural numbers. The assumption that all natural numbers are occupable implies (provably in GB) the consistency of ZF. Since GB is conservative over ZF (provably in GB), it also implies the consistency of GB. To sum up, from Gödel's second incompleteness theorem we have $\mathsf{GB} \nvdash \forall n \mathrm{Ocp}(n)$.

A formula $I(n)$ such that $\mathsf{GB} \vdash I(0)$ and $\mathsf{GB} \vdash \forall n(I(0) \to I(n+1))$ is called a *definable cut*. In GB we can have nontrivial definable cuts, i.e. definable cuts I such that $\mathsf{GB} \nvdash \forall n I(n)$. A nontrivial definable cut violates induction. Therefore, full induction (for all formulas of GB, normal or not) is not provable in GB.

I believe that Vopěnka and Hájek were the first logicians who were aware of the existence of nontrivial definable cuts in GB, and that they discovered them well before 1973, when they published the above construction in [13].

Speaking roughly, one can think of a model (in the sense of the usual logical semantics, i.e. a set model) of GB as a model of ZF where some subsets are interpreted as classes. However, there could be subsets that cannot be accepted as classes without violating the axioms of GB. A model of ZF can contain a "subcollection" that is a subset from the metamathematical point of view, but it is neither a set nor a class in the sense of the model in question. And such a subcollection can even be included in some set of the model. This and similar observations led Vopěnka and Hájek to the notion of semiset and to a new version of axiomatic set theory, the Theory of Semisets TS, see [12]. A *semiset* is defined as a subclass of a set; a semiset is *proper* if it is not a set. TS is weaker than GB (in which no proper semisets exist), but it is still an extension of ZF. Therefore, it is a conservative extension of ZF w.r.t. set sentences. Proper semisets are just a possibility: their existence cannot be proved in TS, and GB can be obtained from TS by adding the axiom every semiset is a set. With semisets, some model constructions including those showing the independence of the continuum hypothesis could be more natural.

Vopěnka and Hájek believed that we (should) have the freedom to work with abstract axiomatic theories. Not only are these theories interesting because one can encounter exciting proofs when thinking inside them, but also they are useful because they provide a safe environment for all classical mathematics. However, Vopěnka and Hájek were finitists in the sense that they (in those times) also believed that on the metamathematical level, i.e. when reasoning *about* the axiomatic system, one has to be careful and use only those tools that are indisputable. In the beginning, the finitistic ideas could have been the reason to prefer working with GB: it is finitely axiomatizable. There is, however, a deeper reason for working with GB or TS, also of a finitist nature. When using the *method of forcing*, as it has developed and as it is now always presented, one works with a model of ZF and with a poset P in it, and with a generic filter on P. The filter is supposed (can by other means be shown) to exist, but cannot exist as an element of the given model. Reasoning about TS could be more economical as to the indisputable tools that are accepted on the metamathematical level: there is no distinction between a model and its outside, because the generic filter can be understood as a semiset existing in the model.

There are two results that may be worth mentioning when talking about set theories, Vopěnka, and Czech logic. In connection with category theory, Vopěnka invented (probably soon after 1964) a large cardinal axiom, now called *Vopěnka's principle*. It has several equivalent formulations, one of which being every proper class of first order structures contains two different members such that one of them can be elementarily embedded to the other. The principle is quite strong, stronger than other popular principles like, say, the existence of measurable cardinals (look at the cover of Kanamori's book [7]). Vopěnka did not really work in the field of large cardinals, and considered them a dead-end in mathematical research. For me personally the fact that he is the author of a widely known result in this area is evidence that he valued a good proof more than a methodological (philosophical, ideological) idea.

While ZF and GB prove the same set sentences, they are not equally efficient. Pavel Pudlák, who does not really belong to this section because he started to work with Petr Hájek later in the 1970s, used proof-theoretic methods to prove a speed-up theorem: eliminating classes from a GB-proof of a set sentence, i.e. constructing a ZF-proof from a given GB-proof of a set sentence, can cause a superexponential increase in the length of the proof.

4 Dissolution of (the classical) seminar

The Russian occupation of Czechoslovakia in August 1968 changed many things. In April 1969 the communist power started what they called 'normalization', i.e. restoration of the totalitarian regime, and these changes had an impact on Vopěnka's group. Vopěnka was identified as a non-cooperating person and lost his say in the administration of the School of Mathematics. He could still do research and had exciting ideas to follow, but his contacts with students were restricted and the department of which he was the head was disbanded. Some people (Jech, Příkrý and Hrbáček) emigrated to the U.S., but some other (younger) people showed interest in working with Vopěnka. The communists did not go so far as to fire Vopěnka from the School of Mathematics, but the threat was there (a few years later they fired Polívka, the philosopher mentioned above).

In this situation Vopěnka told his colleagues that they had learnt everything they could from him, and thus they should not rely on him and find their own topics to work on; he himself would start a new field of research with a new group of collaborators. Vopěnka evidently aimed to stimulate the intellectual development of his colleagues. However, there was also patriotic reasoning behind his decisions: in the times when contacts with the world were violently broken, doing something specifically Czech was his way of maintaining the nation's culture.

For Balcar and Štěpánek the "new" topic consisted in continuing their work in classical set theory. They also published a very influential (Czech) textbook in set theory. Balcar also worked with the topologists Petr Simon and Zdeněk Frolík, while Štěpánek was active in the study programs of the newly created Department of Theoretical Computer Science. Petr Hájek started to work on two different things mentioned below in more detail, the GUHA method on one side, and metamathematics of arithmetic on the other side. Sochor was the only member of the classical seminar whom Vopěnka accepted to his new group (or, as some insiders describe it, who was not sufficiently obedient to follow Vopěnka's advice to find another topic).

Vopěnka's new (the specifically Czech) research field was the Alternative Set Theory, and his new group (new seminar) included for example Karel Čuda, Josef

Mlček, Alena Vencovská, Kateřina Trlifajová (and Sochor). Some axioms and definitions of the *Alternative Set Theory* AST are taken from GB: classes, sets, extensionality, ordinals. In some aspects AST is similar to the theory of finite sets: no limit ordinals exist, and the class N of all natural numbers is a proper class (indeed, it equals the class of all ordinals). The class N contains a proper initial segment FN of all finite natural numbers. The class FN is a proper semiset, numbers in $\mathrm{N} - \mathrm{FN}$ are infinite (nonstandard) natural numbers. There is some freedom in choosing axioms concerning bijections and cardinals. In the most popular and the most natural version, however, for every class X there is a bijection that maps X either on some finite natural number, or on the class FN, or on the entire universe V (of all sets). In the latter case X has cardinality continuum. In this sense there are only two infinite cardinals in AST, the countable infinity and the continuum. Large parts of mathematics can be recovered in AST. However, some parts, like category theory or functional analysis, are problematic or impossible in AST. What can be recovered is *topology*, and what looks very elegant and simple in AST are notions like real numbers or continuous functions, and the differential and integral calculus. This is because it holds that if α is an infinite natural number (a number from $\mathrm{N} - \mathrm{FN}$), then $\frac{1}{\alpha}$ and $-\frac{1}{\alpha}$ are infinitely small number, i.e. *infinitesimals*. Then, for example, a real function f is continuous if $|f(x+\delta) - f(x)|$ is infinitesimal for every infinitesimal δ. Thus AST can be seen as a suitable theory for *nonstandard analysis*. Vopěnka liked to emphasize that quite a substantial amount of mathematics existed before set theory, and that "mathematical analysis" was obtained by shortening the term "mathematical analysis of infinitesimals", used in the times of Leibniz.

The GUHA method, one of the activities of Petr Hájek after the dissolution of the classical seminar, combined statistical and logical tools to automatically discover dependencies in data, with an emphasis on anomalous dependencies shared by only a small fraction of the data. The data were typically medical data (concerning health of patients), but GUHA was of course a general method applicable to data of various kinds. It would now be described as a data-mining method. Hájek worked on it with a group of researchers of different affiliations; they met regularly at a *seminar on applied logic*. The group included Tomáš Havránek, and the book [3] is far from being the only publication about this method.

Another big field of Hájek's research interests lied in *metamathematics of arithmetic*, for which an important initial source of information was Feferman's paper [1]. Besides other things, Hájek studied the interpretability of axiomatic theories. He noticed that the two closely related set theoretic systems ZF and GB differ in interpretability. To look at this in more detail, let $S \rhd T$ be a shorthand for "T is interpretable in S", and let $\mathrm{Intp}(T)$ be the set $\{\, \varphi \,;\, T \rhd (T + \varphi) \,\}$ of all sentences of T whose consistency with T can be shown using an interpretation. It is easy to see that Intp(GB) is a recursively enumerable (r.e.) set. Hájková and Hájek proved in [5] that if A is an r.e. set of set sentences such that no member of A is refutable in ZF (note that this condition is satisfied by the set Intp(GB), and note also that a sentence is not refutable in GB if and only if it is not refutable in ZF), then the set Intp(ZF) $-\, A$ is nonempty. It immediately follows that Intp(ZF) $\neq$ Intp(GB) and that Intp(ZF) is not r.e. Yet another result of Hájek, somewhat less involved but important because it later (in [9]) translated to one of the axioms of interpretability logic, should not be forgotten: if $(S+\varphi) \rhd T$ and $(S+\psi) \rhd T$, then $(S+\varphi\vee\psi) \rhd T$.

Hájek assembled a small group working on this field, which for a while (in 1977) included his wife Marie, Kamila Bendová and myself. The group was soon moved to the Institute of Mathematics (of the Czechoslovak Academy of Sciences), turned into the *seminar on metamathematics of arithmetic*, or simply *Logic Seminar*, and was joined by Pavel Pudlák, whose contribution was enormous. Hájek inspired research in fragments of arithmetic, interpretability of theories and in interpretability logic. This research definitely had an international dimension, which is not quite true about about GUHA or about Vopěnka's AST. For example, the unpublished but often cited Solovay's paper [8] (which used the idea of occupable numbers) answered positively a question asked in [5], i.e. whether Intp(GB) $-$ Intp(ZF) is nonempty. For Petr Hájek, this period ended in the early 1990s when the book [4] appeared. He then moved to another research field: he started an extensive research in *fuzzy logics* and established an active group of researchers working on this topic. The Logic Seminar, with Pudlák and with other distinguished logicians, but without Hájek, and with topics like proof complexity, continues to be very active.

5 Some more motivations and background

In a certain sense, and at least in the earlier period, Vopěnka and Hájek were formalists: they supposed that we work in axiomatic theories and that truth in these theories is determined by their axioms. The first of the following two quotes from the Introduction to [12] (written, just as most of the book, by Hájek) contains "game with symbols", a term that is sometimes taken as evidence of *formalism.*

> "To prove a statement about sets, classes, etc. we appeal to the axioms of the theory in question. On the other hand, statements about statements or proofs are metamathematical statements. [...] In the metamathematical investigation, a theory is to be treated as a purely formal game with symbols."

> "We shall exclude certain methods of proof, such as proof by contradiction, from metamathematical arguments. If we assert that a certain

> proof exists, then we shall always give instructions for constructing it. [. . .] Our methods can be said to be finitistic."

However, the expression "game with symbols" cannot be understood literally. We need axioms to have something reliable, but the choice of axiomatic theories is not arbitrary because these theories are expected to reflect the classical mathematics and serve as an environment for it. When choosing the axioms, we know or at least have a feeling what proofs and concepts should be formalizable in the resulting systems, and the systems should not be unnecessarily strong. The word "finitistic" in the second quotation may be understood as "constructivist", and the whole quotation calls for using intuitionistic logic on the metamathematical level. However, when I asked Hájek around 1977 whether he really thought that intuitionistic logic was the adequate logic for reasoning about axiomatic theories, his answer was no. His position was then roughly as follows. Whenever we have a mathematical proof, we can ask the question what is the formal system in which the proof can be formalized. The system can be weak or not, but "weak" is not the same as "intuitionistic"; we in fact have no good reason to prefer intuitionistic logic to classical logic.

Vopěnka considered finite natural numbers the "visible" part of the universe. The idea that whatever can be observed when looking at the visible world must have at least some continuation behind the horizon is inspired by *phenomenology.* One of Vopěnka's sources of philosophical (phenomenological) thinking was Jiří Polívka. He was a student of Czech philosopher Jan Patočka, who during his studies worked with Heidegger, Husserl and Eugene Fink. A somewhat unrelated remark about Polívka: When he was fired from the School of Mathematics, he accepted a job working at a warehouse, which was not very time consuming, and he could do what he thought was his task—edit the nachlass of Jan Patočka. For Vopěnka, another source of inspiration was *theology.* It sounded like a joke when he lectured about the Greek gods who are not so powerful as the Christian God, but he was serious: he believed that the idea of the variety of infinite cardinal numbers, present in the classical set theory, emerges from theological thinking.

While Petr Hájek was active in the Evangelical Church of Czech Brethren, he separated this activity from his mathematical work. In research, he did not seem to be inspired by religion or by philosophy, and was rather pragmatic when choosing the problems to think about: a good research field for him was one in which it is possible to prove strong and interesting theorems.

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Received March 2017