Rosser sentences and Rosser logics

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Outline

Gödel's sentence, Rosser's sentence

Other prominent self-referential constructions

Provability logic GL and its applications

Rosser logics

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Gödel's sentence: $T \vdash \gamma \equiv \mathrm{Pr}(\overline{\neg\gamma})$.

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Ingredients of independence proofs

(i) The provability predicate $\mathrm{Pr}(x)$ of a theory T *defines* the set $\mathrm{Thm}(T)$ of all provable sentences: $T \vdash \varphi \Leftrightarrow \mathbb{N} \models \mathrm{Pr}(\overline{\varphi})$.

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(ii) Σ-completeness: if σ is a Σ_1-sentence valid in $\mathbb{N}$, then $T \vdash \sigma$.

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The provability predicate $\mathrm{Pr}(x)$ has the form $\exists y \mathrm{Proof}(x, y)$, where $\exists y \mathrm{Proof}(x, y) \in \Delta_1$ is a *proof predicate*, which defines the proof relation: m is a proof of φ iff $\mathbb{N} \models \mathrm{Proof}(\overline{\varphi}, \overline{m})$.

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$$T \vdash \rho \equiv \exists y (\mathrm{Proof}(\overline{\neg\rho}, y) \; \& \; \forall v < y \, \neg \mathrm{Proof}(\overline{\rho}, v))$$

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Are different ways to write the sentence.

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$$T \vdash \rho \equiv \mathrm{Pr}(\overline{\neg\rho}) \preceq \mathrm{Pr}(\overline{\rho}),$$ (witness comparison symbols $\preceq, \prec$)

Are different ways to write the sentence.

Independence of Rosser's sentence

A canonical proof

Let $T \vdash \rho$ or $T \vdash \neg\rho$. Then $\mathbb{N} \models \mathrm{Pr}(\overline{\rho})$ or $\mathbb{N} \models \mathrm{Pr}(\overline{\neg\rho})$.

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$\mathbb{N} \models \mathrm{Pr}(\overline{\neg\rho}) \preceq \mathrm{Pr}(\overline{\rho})$ and
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In the left case we have $\mathbb{N} \models \mathrm{Pr}(\overline{\neg\rho})$, $T \vdash \neg\rho$ and $T \vdash \rho$.

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In the right case we have $\mathbb{N} \models \mathrm{Pr}(\overline{\rho})$ and $T \vdash \rho$. Inside T, $\mathrm{Pr}(\overline{\rho}) \prec \mathrm{Pr}(\overline{\neg\rho})$ yields $\neg(\mathrm{Pr}(\overline{\neg\rho}) \preceq \mathrm{Pr}(\overline{\rho}))$, which is $\neg\rho$.

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A variant proof

If $T \vdash \rho$, then $T \vdash \mathrm{Proof}(\overline{\rho}, \overline{m})$ for some m, and
$T \vdash \neg\mathrm{Proof}(\overline{\neg\rho}, \overline{n})$ for every n.

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A canonical proof

Let $T \vdash \rho$ or $T \vdash \neg\rho$. Then $\mathbb{N} \models \mathrm{Pr}(\overline{\rho})$ or $\mathbb{N} \models \mathrm{Pr}(\overline{\neg\rho})$. Thus

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If $T \vdash \rho$, then $T \vdash \mathrm{Proof}(\overline{\rho}, \overline{m})$ for some m, and $T \vdash \neg\mathrm{Proof}(\overline{\neg\rho}, \overline{n})$ for every n. Then $T \vdash \mathrm{Pr}(\overline{\rho}) \prec \mathrm{Pr}(\overline{\neg\rho})$,

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A variant proof

If $T \vdash \rho$, then $T \vdash \mathrm{Proof}(\overline{\rho}, \overline{m})$ for some m, and $T \vdash \neg\mathrm{Proof}(\overline{\neg\rho}, \overline{n})$ for every n. Then $T \vdash \mathrm{Pr}(\overline{\rho}) \prec \mathrm{Pr}(\overline{\neg\rho})$, hence $T \vdash \neg(\mathrm{Pr}(\overline{\neg\rho}) \preceq \mathrm{Pr}(\overline{\rho}))$, which means $T \vdash \neg\rho$.
Similarly, if $T \vdash \neg\rho$, then $T \vdash \mathrm{Proof}(\overline{\neg\rho}, \overline{m})$ for some m, and $T \vdash \neg\mathrm{Proof}(\overline{\rho}, \overline{n})$ for every n. Then $T \vdash \rho$.

Aspects and remarks

1. Comparison of Gödel's and Rosser's sentences: some parts of the above reasoning are, but some are not formalizable in the theory itself:
   $T \vdash \mathrm{Con}(T) \rightarrow \neg\mathrm{Pr}(\overline{\neg\gamma})$,
   $T \nvdash \mathrm{Con}(T) \rightarrow \neg\mathrm{Pr}(\overline{\gamma})$,
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2. Comparison of the canonical and the variant proof: the variant proof is less demanding on resources: Σ-completeness is applied to simpler sentences $(\mathrm{Proof}(\ldots, \overline{n})$ or $\neg\mathrm{Proof}(\ldots, \overline{n}))$, but not to $\mathrm{Pr}(\ldots)$); some induction is involved in the canonical proof.

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3. Uniqueness: the self-reference guarantees the existence of certain sentence, but does not say that it is unique.

Some other self-referential sentences

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Mostowski's flexible formula

is a formula $\varphi(x)$ such that $T \cup \{\, \pm\varphi(\overline{0}), \pm\varphi(\overline{1}), \pm\varphi(\overline{2}), \ldots \}$ is consistent for every choice of pluses and minuses.

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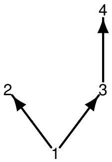
Hájková-Hájek

$\mathrm{PA} \vdash \mu \equiv \forall y(\mathrm{Con}(\mathrm{PA}{\restriction} y + \overline{\mu}) \rightarrow \neg\alpha(\overline{\mu}, y))$, where $\mathrm{PA}{\restriction} y$ are axioms of PA less than y, and $\alpha(x, y) \in \Delta_0$ defines a set A of consistent sentences. Then $\mu \notin A$, but $\mathrm{PA} \rhd \mathrm{PA} + \mu$.

Self-referential sentences (continued)

Embedding a Kripke model to PA (Solovay)

Let $k = \langle W, R \rangle$, $W = \{1, .., n\}$. Put $S(i) = \{\, j \,;\, i\, R\, j \,\}$. Example:



$S(1) = \{2, 3, 4\}$, $S(3) = \{4\}$, $S(2) = S(4) = \emptyset$.

Then consistent sentences $\lambda_1, .., \lambda_n$ such that

$$\text{PA} \vdash \lambda_i \;\rightarrow\; \bigwedge_{j \in S(i)} \neg\mathrm{Pr}(\overline{\neg\lambda_j}) \;\text{ and }\; \text{PA} \vdash \lambda_i \;\rightarrow\; \mathrm{Pr}(\overline{\bigvee_{j \in S(i)} \lambda_j}).$$

and furthermore $\lambda_i \rightarrow \neg\lambda_j$ for $i \neq j$ are constructed using *plural but finite* self-reference (solvability of n equations for n unknown sentences).

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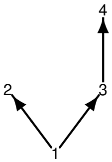

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and furthermore $\lambda_i \rightarrow \neg\lambda_j$ for $i \neq j$ are constructed using *plural but finite* self-reference (solvability of n equations for n unknown sentences).

A self-referential equation $\vdash \varphi \equiv \psi(\overline{\varphi})$ is *Gödelian* if ψ is built up using connectives and Pr only.

Provability logic GL

is a modal propositional logic with axioms and rules as follows:

A1: all propositional tautologies,

A2: $\Box(A \to B) \to (\Box A \to \Box B)$, MP: $A \to B, A \,/\, B$,

A3: $\Box A \to \Box\Box A$, Nec: $A \,/\, \Box A$.

A4: $\Box(\Box A \to A) \to \Box A$,

Arithmetic semantics of GL

Arithmetic valuation v is a function from modal formulas to sentences of arithmetic that preserves logical connectives and translates $\Box$ to Pr (atoms are sent to any sentences).

A formula A is a PA-*tautology* if $\mathrm{PA} \vdash v(A)$ for every translation v.

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A1: all propositional tautologies,

A2: $\Box(A \to B) \to (\Box A \to \Box B)$, MP: $A \to B, A \,/\, B$,

A3: $\Box A \to \Box\Box A$, Nec: $A \,/\, \Box A$.

A4: $\Box(\Box A \to A) \to \Box A$,

Arithmetic semantics of GL

Arithmetic valuation v is a function from modal formulas to sentences of arithmetic that preserves logical connectives and translates $\Box$ to Pr (atoms are sent to any sentences).
A formula A is a PA-*tautology* if PA $\vdash v(A)$ for every translation v.

Examples

$\Box p \to p$ is not a PA-tautology: for Gödel's sentence γ we have PA $\nvdash \mathrm{Pr}(\overline{\neg\gamma}) \to \neg\gamma$. A tautology: $\neg\Box\bot \to \neg\Box\neg\Box\bot$.

Properties and applications

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   if PA $\vdash \kappa \equiv \mathrm{Pr}(\overline{\kappa})$, then PA $\vdash \gamma \equiv 0 = 0$;
   if PA $\vdash \lambda \equiv \mathrm{Pr}(\overline{\lambda}) \rightarrow \kappa$, then PA $\vdash \lambda \equiv \mathrm{Pr}(\overline{\kappa}) \rightarrow \kappa$
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   (can be verified by proving $\Box q \rightarrow q \equiv \Box(\Box q \rightarrow q) \rightarrow q$ in GL); etc.
4. Impossibility of symmetrically independent Gödelian sentence: no solution φ of a Gödelian equation $\vdash \varphi \equiv \psi(\overline{\varphi})$ satisfies PA $\vdash \mathrm{Con}(\mathsf{PA}) \rightarrow \neg\mathrm{Pr}(\overline{\varphi})\ \&\ \neg\mathrm{Pr}(\overline{\neg\varphi})$.

The logic R of Guaspari and Solovay

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The theory R^- is obtained by adding to GL the axioms about witness comparison:

B1: $\Box A \preceq \Box B \;\rightarrow\; \Box A$,

B2: $\Box A \preceq \Box B \;\&\; \Box B \preceq \Box C \;\rightarrow\; \Box A \preceq \Box C$,

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B4: $\Box A \vee \Box B \;\rightarrow\; \Box A \preceq \Box B \vee \Box B \prec \Box A$,

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Kripke semantics: formulas with $\preceq, \prec$ as the outermost symbol are treated as atoms (with the restrictions given by the axioms).

The logic R (continued)

Arithmetic semantics: the modalities $\preceq, \prec$ translate to $\preceq, \prec$:
$v(\Box A \preceq \Box B) = \mathrm{Pr}(\overline{v(A)}) \preceq \mathrm{Pr}(\overline{v(B)})$,
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Completeness theorem

If R $\nvdash A$, then there exists a valuation v (i.e. a choice of a proof predicate and values of atoms) such that PA $\nvdash v(A)$.

Some applications of the logic R

1. R can prove Rosser's theorem in the form
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2. The Solovay's plural self-reference construction (concerning completeness of GL) can be formalized in GL (I believe).
3. Consider the formula $A = \Box(p \equiv \Box\neg p \preceq \Box p) \ \& \ \Box(q \equiv \Box\neg q \preceq \Box q) \to \Box(p \equiv q)$. Use Kripke semantics to show that this formula is not provable. Take the arithmetic counterexemple, i.e. sentences $v(p)$ and $v(q)$ and a proof predicate Pr such that $\mathbb{N} \models \mathrm{Pr}(\overline{p \equiv \mathrm{Pr}(\overline{\neg v(p)}) \preceq \mathrm{Pr}(\overline{v(p)})})$, $\mathbb{N} \models \mathrm{Pr}(\overline{q \equiv \mathrm{Pr}(\overline{\neg v(q)}) \preceq \mathrm{Pr}(\overline{v(q)})})$, but $\mathbb{N} \not\models \mathrm{Pr}(\overline{v(p) \equiv v(q)})$. Then $v(p)$ and $v(q)$ are non-equivalent.

Alternative Rosser logics. Why?

Look at the sentence of Hájková and Hájek: a number y such that $\neg\mathrm{Con}(\mathsf{PA}\restriction y + \overline{\mu})$ can be understood as a generalized proof of $\neg\mu$, and the formula $\neg\mathrm{Con}(\mathsf{PA}\restriction y + x)$ can be seen as a sort of proof predicate.

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$\mathsf{PA} \vdash \varphi \equiv \exists y(\neg\mathrm{Con}(\mathsf{PA}{\restriction}\, y + \overline{\varphi})\ \&\ \mathrm{Con}(\mathsf{PA}{\restriction}\, y + \overline{\varphi}))$
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W: $\Box A \rightarrow \Box(\neg B \rightarrow \Box A \prec \Box B)$, $\qquad A \,/\, \neg B \rightarrow \Box A \prec \Box B$

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