Exercises to *Properties of axiomatic theories* (January 15, 2024)

Exercises

1. Let P and Q be unary and R a binary predicate. Prove that the following sentences are logically valid, but reverting the outermost implication yields (in all cases) a formula that is not logically valid:
   $\exists x(P(x) \,\&\, Q(x)) \rightarrow \exists x P(x) \,\&\, \exists x Q(x)$,
   $\forall x P(x) \vee \forall x Q(x) \rightarrow \forall x(P(x) \vee Q(x))$,
   $\exists x \forall y R(x,y) \rightarrow \forall y \exists x R(x,y)$,
   $\forall x(P(x) \rightarrow Q(x)) \rightarrow (\forall x P(x) \rightarrow \forall x Q(x))$,
   $\forall x(P(x) \rightarrow Q(x)) \rightarrow (\exists x P(x) \rightarrow \exists x Q(x))$.

2. Which of $\forall x(P(x) \rightarrow \forall y P(y))$, $\exists x(P(x) \rightarrow \forall y P(y))$ and $\exists x(\exists y P(y) \rightarrow P(x))$ are logically valid sentences?

3. For every sentence from the previous two exercises that is logically valid prove its provability in the Hilbert-style calculus. Use tautological consequences and the fact that all tautologies are provable, but avoid using the predicate completeness theorem (otherwise there would be nothing to do).

4. Show that $\Delta, \psi \models \varphi$ if and only if $\Delta \models \psi \rightarrow \varphi$ for any formulas φ and ψ and any set Δ of formulas.

5. Theories T and S are *equivalent* if every axiom of S is a consequence of T, and at the same time every axiom of T os a consequence of S. Prove that T and S are equivalent if and only if they have the same models (that is, every model of T is a model of S and vice versa).

6. Let φ be a formula in a language L. Consider the conditions (i) there exists a number n and terms $t_1, \ldots, t_n$ of L such that $\varphi_x(t_1) \vee \ldots \vee \varphi_x(t_n)$ is a logically valid formula, and (ii) the formula $\exists x \varphi$ is logically valid. Show that (ii) is a consequence of (i) but (ii) $\Rightarrow$ (i) is not necessarily true.

   Hint. Let L be $\{P\}$ and let φ be the formula $P(x) \rightarrow \forall v P(v)$. Since there are no function symbols, $t_1, \ldots, t_n$ must be variables, say $z_1, \ldots, z_n$ with possible repetitions. However, no disjunction of the form $\bigvee_i (P(z_i) \rightarrow \forall v P(v))$ is logically valid.

7. The claim that if φ is open, then conditions (i) and (ii) in the previous exercise are equivalent is true and is known as the Hilbert-Ackermann theorem. This theorem was omitted in the course. Explain that one term would not be enough: if φ is an open formula in L and $\exists x\varphi$ is logically valid, then there may not exist a single term t of L such that $\varphi_x(t)$ is logically valid.

   Hint. Pick the language $\{P, F\}$ containing a unary predicate and a unary function and consider the formula $P(x) \vee \neg P(F(x))$. The term t must have the form $F^{(m)}(z)$ where z is a variable.

8. For the formula φ from the above hint find an n and terms $t_1, \ldots, t_n$ of L such that $\varphi_x(t_1) \vee \ldots \vee \varphi_x(t_n)$ is logically valid.

9. Let the language of T be $\{\in\}$ and let its axioms be

   $\forall x\forall y(\forall v(v \in x \equiv v \in y) \rightarrow x = y)$,

   $\exists x\forall v\neg(v \in x)$,

   $\forall x\forall y\exists z\forall v(v \in x \vee v = y \rightarrow v \in z)$.

   (a) Use finite models to show that $\forall x(x \notin x)$ and $\neg\exists x\forall v(v \in x)$ cannot be proved in T.

   (b) Prove that none of the axioms of T is provable from the remaining two.

10. Let T be a theory with an empty language and no axioms. Describe all models of T. Find an extension S of T formulated in the same (empty) language such that S is consistent and has no finite models.

11. For each of the structures $\langle \mathrm{N}, < \rangle$, $\langle \mathrm{Z}, < \rangle$ a $\langle \mathrm{Q}, < \rangle$ find a sentence that is valid in it but is not valid in the remaining two structures. Can also the structures $\mathbb{R}$ and $\mathbb{Q}$ be distinguished by the validity of some sentence? And what about $\langle \mathrm{Z}, + \rangle$ and $\langle \mathrm{Q}, + \rangle$?

12. Show that the structures $\langle \mathrm{R}, < \rangle$ a $\langle \mathrm{R} - \{0\}, < \rangle$ are not isomorphic. Prove that they are elementarily equivalent.

    Hint. Every nonempty set bounded from above has the least upper bound in $\langle \mathrm{R}, < \rangle$. This is not true about $\langle \mathrm{R} - \{0\}, < \rangle$. The two structures are models of the same complete theory.

13. Use Vaught's test to show that the theory S from Exercise 10 is complete.

14. Show that if T is equivalent (in the sense of Exercise 5) to some finite set of sentences, then it is equivalent to its own finite subset. Conclude that the theory S from Exercise 10 is not finitely axiomatizable. The theory SUCC is not finitely axiomatizable either.

15. Prove that if a class $\mathcal{C}$ of structures for a language L is axiomatizable and its complement $-\mathcal{C}$ (that is, the class of all structures for L that are not in $\mathcal{C}$) is axiomatizable as well, then both $\mathcal{C}$ and $-\mathcal{C}$ are finitely axiomatizable.

16. Show that the class of all connected graphs, understood as structures for a language with a binary predicate as the only symbol, is not axiomatizable.

17. Consider the class of all structures $\langle D, P\rangle$ for a language with a unary predicate such that both P and $D - P$ are infinite. Prove that this class is axiomatizable. Is it finitely axiomatizable? For which κ is the corresponding theory κ-categorical?

18. Show that the theory whose axioms are Q1–Q5 is a conservative extension of the theory with axioms Q1–Q3.

19. Use the same method to prove that adding the axioms Q4 and Q5 to SUCC yields a conservative extension of SUCC. Prove the same using the following fact: *every consistent extension of a complete theory is a conservative extension*. Explain that this fact is true. Prove that also $\mathrm{Th}(\langle \mathrm{N}, +, 0, \mathrm{s}\rangle)$ is a conservative extension of SUCC. Explain that this last claim cannot be proved using the method from the previous exercise: no expansion of the structure $\langle \mathrm{N}, 0, \mathrm{s}\rangle + \langle \mathrm{Z}, \mathrm{s}\rangle$ is a model of $\mathrm{Th}(\langle \mathrm{N}, +, 0, \mathrm{s}\rangle)$.

   Hint. There is no realization of the symbol $+$ such that the sentences $\forall x \exists y(x = y + y \ \vee\ x = \mathrm{S}(y + y))$ and $\forall x \forall y \forall z(z + x = z + y \ \rightarrow\ x = y)$ are valid.

20. Let γ be the sentence $\forall x(\mathrm{S}(\mathrm{S}(\mathrm{S}(x))) = x \ \rightarrow\ \exists y(((y + x) + x) + x = y))$. Prove it in Q. Finish a proof, invented by Jan Urbánek, that Q is not a conservative extension of the theory Q1–Q5.

   Hint. To prove γ in Q, work with $y = x \cdot x$. To show Q1–Q5 $\nvdash \gamma$, add three nonstandard elements a, b and c to the structure $\mathbb{N}$ and define that $\mathrm{S}^{\mathcal{M}}(a) = b$, $\mathrm{S}^{\mathcal{M}}(b) = c$ and $\mathrm{S}^{\mathcal{M}}(c) = a$. Define $+^{\mathcal{M}}$ so that it extends $+^{\mathbb{N}}$ and satisfies $a +^{\mathcal{M}} a = b +^{\mathcal{M}} a = c$ and $c +^{\mathcal{M}} a = a$.

21. Put $M = \mathrm{N} \cup \{a, b\}$ and let a successor function on M be defined so that the successor of a standard number n, the element a and the element b are $n + 1$, b and a respectively. Show that there are (multiple) ways how to define addition and multiplication on M so that the resulting structure $\mathcal{M}$ is a model of Q.

22. Find out which of the following sentences are provable in Q:

   $\forall x(x \leq x)$ $\qquad$ $\forall x \forall y(x + y = 0 \rightarrow x = 0 \ \&\ y = 0)$

   $\forall x(x \leq 0 \rightarrow x = 0)$ $\qquad$ $\forall x \forall y(x \leq y \ \equiv\ \mathrm{S}(x) \leq \mathrm{S}(y))$

$\forall x(0 \leq x)$ $\quad$ $\forall x \forall y(x < y \to x < \mathrm{S}(y))$
$\forall x(0 \cdot x = 0)$ $\quad$ $\forall x \forall y(\mathrm{S}(x) < y \to x < y)$
$\forall x(x \cdot \overline{1} = x)$ $\quad$ $\forall x \forall y(x \cdot y = 0 \to x = 0 \vee y = 0)$
$\forall x \forall y \exists z(x \leq z \ \& \ y \leq z)$ $\quad$ $\forall x(x \leq \overline{1} \to x = 0 \vee x = \overline{1})$
$\forall x \neg(x < x)$ $\quad$ $\forall x \forall y \forall z((z + y) + x = z + (y + x))$.
$\forall x \forall y(x \leq y \to x < y \vee x = y)$

Hint. Unprovability can be proved by a suitable choice of operations in the previous exercise, and just two models are sufficient.

23. Show that every natural number is a definable element of $\langle \mathrm{N}, < \rangle$. Furthermore, let R be the relation $\{ [a, b] \, ; \, |a - b| = 1 \}$. Prove that every natural number is a definable element of $\langle \mathrm{N}, R \rangle$.

24. Use Post's theorem to prove that if $X \subseteq \mathrm{N}^q$ and $Y \subseteq \mathrm{N}^q$ are *RE* sets such that $X \cup Y$ is recursive and $X \cap Y = \emptyset$, then both X and Y are recursive.

25. Show that if $f : \mathrm{N} \to \mathrm{N}$ is a strictly increasing recursive function, then its range is recursive.

26. Prove that if $R \subseteq \mathrm{N}^2$ is an equivalence having only finitely many classes (equivalence classes) and is *RE*, then R must be recursive.

    Hint. Let $A_1 \ldots, A_n$ be a list of all equivalence classes of R. Explain in detail the following facts. Every A_i is *RE*, its complement is *RE* as well, and R can be defined in terms of $A_1 \ldots, A_n$ via a recursive condition.

27. A function $f : \mathrm{N}^q \to \mathrm{N}$ defined as $f(n_1, \ldots, n_q) = 1$ for $[n_1, \ldots, n_q] \in A$ and $f(n_1, \ldots, n_q) = 0$ for $[n_1, \ldots, n_q] \notin A$ is called *characteristic function* of a set $A \subseteq \mathrm{N}^q$. It is clear that if $\varphi(\underline{x}, y)$ defines the graph of a characteristic function f of A and is Σ_1, then $\varphi(\underline{x}, \overline{1})$ defines A and $\varphi(\underline{x}, 0)$ defines $-A$. Thus $A \in \Delta_1$. Show that the converse is also true: the characteristic function of a recursive set must be recursive.

28. Show that if A is an r-ary recursive (or *RE*, or Π_1) condition and $g_1, \ldots, g_r$ are recursive functions of q variables, then $\{[n_1, \ldots, n_q]; A(g_1(\underline{n}), \ldots, g_r(\underline{n})\}$ is recursive (or *RE*, or Π_1 respectively). Put otherwise, substituting recursive functions into a Δ_1 (or *RE*, or Π_1) condition yields a Δ_1 (or *RE*, or Π_1) condition.

29. Prove that $\mathrm{Thm}(T) = \bigcap \{ \mathrm{Thm}(S) \, ; \, S$ is a complete extension of $T \}$ holds for any theory T. Conclude that if the number of all complete extensions of T formulated in the same language is finite, and all of them are decidable, then T is decidable. It follows that the theory obtained from DNO by removing the axioms postulating the existence of the greatest and the least individual is decidable.

30. Let T be a recursively axiomatizable extension of Q such that T is formulated in the arithmetic language and is sound (in the sense that $\mathbb{N} \models T$). Find out whether the following claims are true.
(a) If φ and ψ are sentences such that $T \vdash \varphi \vee \psi$, then $T \vdash \varphi$ or $T \vdash \psi$.
(b) if φ and ψ are Σ_1-sentences such that $T \vdash \varphi \vee \psi$, then $T \vdash \varphi$ or $T \vdash \psi$.

Hint. In (a), use Gödel's first incompleteness theorem. In (b) apply the Σ-completeness theorem separately to φ and to ψ.

31. In the same situation find out whether the following claims are true.
(a) If $\exists x \varphi(x)$ is an arithmetic sentence such that $T \vdash \exists x \varphi(x)$, then there exists a number n such that $T \vdash \varphi(\overline{n})$.
(b) If $\exists x \varphi(x)$ is an arithmetic sentence such that $T \vdash \exists x \varphi(x)$ and $\varphi \in \Delta_0$, then there exists a number n such that $T \vdash \varphi(\overline{n})$.

Hint. In (a) pick a formula $\psi(y) \in \Delta_0$ such that $\mathbb{N} \models \forall y \psi(y)$ and $T \nvdash \forall y \psi(y)$. The existence of a formula like that is guaranteed by Gödel's first incompleteness theorem. Then consider the sentence $\exists x \forall y(\psi(y) \vee \neg\psi(x))$.